1. Determine whether the series converges or diverges. If it converges, does it converge conditionally or absolutely?

\[ \sum_{n=1}^{\infty} \frac{n^2(-3)^n}{(2n+1)!} \]

**Solution.**

\[ a_{n+1} = \frac{(n+1)^2(-3)^{n+1}}{(2(n+1)+1)!} = \frac{(n+1)^2(-3)^{n+1}}{(2n+3)!} \]

We can use ratio test:

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(n+1)^2 \cdot 3^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{n^2 \cdot 3^n} \]

\[ = \lim_{n \to \infty} \frac{(n+1)^2}{n^2} \cdot \frac{3^{n+1}}{3^n} \cdot \frac{(2n+1)!}{(2n+3)(2n+2)(2n+1)!} \]

\[ = \lim_{n \to \infty} \frac{3(n+1)^2}{n^2(2n+3)(2n+2)} = 0 \quad \text{(Because top has highest power } n^2 \text{ and bottom has highest power } n^4) \]

Therefore \( \sum a_n \) converges.

2. For which values of \( p \) does the following series converge? (If it converges only conditionally, we also say it converges.)

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^p} \]

**Solution.** The above series converges when \( p > 0 \)

diverges when \( p \leq 0 \).

When \( p \geq 0 \):

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^p} = \sum_{n=1}^{\infty} (-1)^n n^{-p} \text{ where } f(p) > 0. \]

so you will have for example, \( \sum_{n=1}^{\infty} (-1)^n n^2, \sum_{n=1}^{\infty} (-1)^n n^3 \ldots \)

which diverges.

You can verify that it diverges because \( \lim_{n \to \infty} a_n \neq 0 \).

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• When $p = 0$: you have $\sum_{n=1}^{\infty} (-1)^n$, which diverges.

• When $p > 0$: \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^p} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^p} \] is a strictly alternating series.

   Now $\frac{1}{n^p}$ decreases and converges to 0.

   So by alternating series test, the series converges.

• When $0 < p < 1$: for example, \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{0.5}} \text{ or } \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1.5}} \]

   because $p < 1$, \[ \sum \left| \frac{(-1)^n}{n^p} \right| = \sum \frac{1}{n^p} \] diverges.

   So the series is only conditionally convergent.

• When $p = 1$: similar to above, it is conditionally convergent.

• When $p > 1$: \[ \sum \left| \frac{(-1)^n}{n^p} \right| = \sum \frac{1}{n^p} \] converges,

   so the series is absolutely convergent.