1. Use integral test to determine whether the series converges or diverges:

$$\sum_{k=2}^{\infty} \frac{1}{k \ln k}$$

Solution. Let $f(x) = \frac{1}{x \ln x}$. It is continuous on $[2, \infty)$, positive, decreasing, so integral test applies. For integration, use the substitution: $u = \ln x$, $du = \frac{1}{x} dx$.

$$\begin{aligned}
\int_2^\infty \frac{1}{x \ln x} dx &= \lim_{T \to \infty} \int_2^T \frac{1}{x \ln x} dx \\
&= \lim_{T \to \infty} \int_{\ln 2}^{\ln T} \frac{1}{u} du \\
&= \lim_{T \to \infty} \left[\ln u \Big|_{\ln 2}^{\ln T} \right] \\
&= \lim_{T \to \infty} [\ln(\ln T) - \ln(\ln 2)] \\
&= \infty
\end{aligned}$$

Therefore the improper integral $\int_2^\infty \frac{1}{x \ln x} dx$ diverges. Consequently, the series $\sum_{k=2}^\infty \frac{1}{k \ln k}$ diverges.

2. Use the comparison or limit comparison test to determine whether the following series converges or diverges:

(1)

$$\sum_{n=1}^{\infty} \frac{n+3}{4n^3+3}$$

Solution. By looking at the dominant terms (highest powers on top and bottom of the fraction), we compare

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n+3}{4n^3+3}$$

to

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n}{4n^3} = \sum_{n=1}^{\infty} \frac{1}{4n^2}$$

Since

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\frac{n+3}{4n^3+3}}{\frac{1}{4n^2}} = \lim_{n \to \infty} \frac{(n+3)4n^2}{4n^3+3} = \lim_{n \to \infty} \frac{4n^3+12n^2}{4n^3+3} = \lim_{n \to \infty} \frac{4 + \frac{12n^2}{n^3}}{4 + \frac{3}{n^3}} = \frac{4+0}{4+0} = 1 > 0$$

And since $\sum b_n$ converges by p-series ($p = 2$), we conclude that $\sum a_n$ **converges by the Limit Comparison Test.**
Note.

- $\frac{n+3}{4n^3+3} \geq \frac{n}{4n^3}$, so you can't directly use comparison test with $b_n = \frac{n}{4n^3}$. However, it is true that $\frac{n+3}{4n^3+3} \leq \frac{n}{n^3}$ for large n, so you can also use comparison test with $b_n = \frac{n}{n^3}$.
- For Limit Comparision Test, we just requite $\lim_{n \to \infty} \frac{a_n}{b_n} = c$ where c is a positive constant. Don't compare c to 1 (that's Root/Ratio test).

(2)

$$\sum_{k=1}^{\infty} \frac{4}{3+5^k}$$

Solution. By looking at the dominant terms, we compare

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{4}{3+5^k}$$

to

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{4}{5^k}$$

Since

$$a_k \leq b_k$$

And since $\sum b_k$ converges because it is a **geomertic series with** $\mathbf{r} = \frac{\mathbf{1}}{\mathbf{5}}$, we conclude that $\sum a_k$ **converges by the Comparison Test.**