Selected Old Quest Problems for Differential Equations

9.3

For the differential equation
\[ \frac{5}{x} \frac{dy}{dx} + \frac{5}{xy^4} = 0, \ (x, y > 0), \]

(1) Find the general solution.
(2) If \( y(1) = 2 \), find the particular solution.
(3) Find the value of \( y(e) \).

9.4

1. A drug becomes ineffective at a rate proportional to the amount still present. In other words,
\[ \frac{dP}{dt} = kP \text{ for some } k < 0. \]
Half of the drug is effective at exactly \( t = 13 \) day.
(1) Find \( k \).
(2) What is \( t \) when the drug has 90% ineffective ingredients? (i.e. \( P(t) = 10\% P(0) \))

2.* (Set up an equation by yourself)
Scientists began studying the elk population in Yellowstone Park in 1990 when there were 600 elk. They determined that \( t \) years after the study began the population size, \( P(t) \), was increasing at a rate proportional to \( 1500 - P(t) \).
Given that the population was 1300 in year 2000,
(1) Set up a differential equation for \( P(t) \);
(2) Using the given values of elk population, find the particular solution to differential equation;
(3) Estimate the size of the elk population in year 2010 (need to use a calculator).
**Series Convergence/Divergence Flow Chart**

**TEST FOR DIVERGENCE**

Does \( \lim_{n \to \infty} a_n = 0? \)

\[ \sum a_n \text{ Diverges} \]

**p-SERIES**

Does \( a_n = \frac{1}{n^p}, \ n \geq 1? \)

\[ \text{YES} \]

\[ \text{NO} \]

\[ \text{YES} \]

\[ \text{NO} \]

**GEOMETRIC SERIES**

Does \( a_n = ar^{n-1}, \ n \geq 1? \)

\[ \text{YES} \]

\[ \text{NO} \]

**ALTERNATING SERIES**

Does \( a_n = (-1)^n b_n \) or \( a_n = (-1)^{n-1} b_n, \ b_n \geq 0? \)

\[ \text{YES} \]

\[ \text{NO} \]

**TELESCOPING SERIES**

Do subsequent terms cancel out previous terms in the sum? May have to use partial fractions, properties of logarithms, etc. to put into appropriate form.

\[ \text{YES} \]

\[ \text{NO} \]

**TAYLOR SERIES**

Does \( a_n = \frac{f^{(n)}(a)}{n!}(x-a)^n? \)

\[ \text{YES} \]

\[ \text{NO} \]

**Try one or more of the following tests:**

**COMPARISON TEST**

Pick \( \{b_n\}. \) Does \( \sum b_n \) converge?

\[ \text{YES} \]

\[ \text{NO} \]

**LIMIT COMPARISON TEST**

Pick \( \{b_n\}. \) Does \( \lim_{n \to \infty} \frac{a_n}{b_n} = c > 0 \) \( c \) finite \& \( a_n, b_n > 0? \)

\[ \text{YES} \]

\[ \text{NO} \]

**INTEGRAL TEST**

Does \( a_n = f(n), \ f(x) \) is continuous, positive \& decreasing on \( [a, \infty)? \)

\[ \text{YES} \]

\[ \text{NO} \]

**RATIO TEST**

Is \( \lim_{n \to \infty} |a_{n+1}/a_n| \neq 1? \)

\[ \text{YES} \]

\[ \text{NO} \]

**ROOT TEST**

Is \( \lim_{n \to \infty} \sqrt[n]{|a_n|} \neq 1? \)

\[ \text{YES} \]

\[ \text{NO} \]
1 Selected Old Quest Problems for Series

1.1 11.2

1. If the n-th partial sum $S_n$ of an infinite series

$$S_n = 8 - \frac{n}{3^n}$$

find $a_n$. Also, does the series $\sum_{0}^{\infty} a_n$ converge? What does it converge to?

Answer: $a_0 = S_0 = 8$, $a_n = S_n - S_{n-1} = \frac{2n-3}{3^n}$ for $n \geq 1$.

$\sum_{0}^{\infty} a_n = \lim_{n \to \infty} S_n = 8$.

2. Determine whether the sequence converges:

$$a_n = \frac{n + (-3)^n}{5^n}$$

Does the series converge?

$$\sum_{n=0}^{\infty} \frac{n + (-3)^n}{5^n}$$

Answer: Both yes. (Hint: $\sum \frac{(-3)^n}{5^n}$ converges. For $\sum \frac{n}{5^n}$, use comparison test/ratio test to show it converges.)

1.2 11.3

Determine whether the series converges:

$$\sum_{k=0}^{\infty} \frac{1}{k(\ln(2k))^2}$$

Answer: Yes.

1.3 11.4

1. Determine whether the series converges:

$$\sum_{n=0}^{\infty} \frac{\sin(n)}{n^2}$$

Answer: Yes ($|\sin(n)| \leq 1$)

2. Determine whether the series converges. If converges, conditionally or absolutely:

$$\sum_{n=0}^{\infty} (-1)^n \sin\left(\frac{1}{3n}\right)$$

Answer: converges conditionally. (For $\sum \sin\left(\frac{1}{3n}\right)$, compare to $\sum \frac{1}{3n}$.)
1.4  11.5.4
Determine whether the series
\[
\sum_{n=0}^{\infty} \frac{4}{\sqrt{n+2}} \cos(n\pi)
\]
converges or diverges.
Answer: converges.

1.5  11.6.4
Decide whether the series
\[
\sum_{n=0}^{\infty} 2^n \left( \frac{n-2}{n} \right)^{n^2}
\]
converges or diverges.
Answer: converges.

1.6  11.7.5
Determine which, if either, of the series
1. \[\sum_{n=1}^{\infty} \frac{(-1)^n+2}{\sqrt{n}}\]
2. \[\sum_{k=3}^{\infty} \frac{(-1)^k}{2k \ln(k+3)}\]
are conditionally convergent.
Answer: Both.

1.7  11.8.2
If the series
\[
\sum_{n=0}^{\infty} c_n 4^n
\]
is convergent, which of the following statements must be true without further restrictions on \(c_n\).
1. \[\sum_{n=0}^{\infty} c_n (-4)^n\] is convergent
2. \[\sum_{n=0}^{\infty} c_n (-4)^n\] is divergent
3. \[\sum_{n=0}^{\infty} c_n (-3)^n\] is convergent
4. \[\sum_{n=0}^{\infty} c_n (-3)^n\] is divergent
Answer: 3 is true.
### 1.8 11.9.4

Find a power series representation for the function

\[ f(x) = \ln(7 - x) \]

**Answer:**

\[ f(x) = \ln(7) - \sum_{n=1}^{\infty} \frac{x^n}{n!} \]

We can either use the known power series representation

\[ \ln(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \]

or the fact that

\[
\ln(1 - x) = -\int_0^x \frac{1}{1 - s} \, ds \\
= -\int_0^x \left( \sum_{n=0}^{\infty} s^n \right) \, ds \\
= -\sum_{n=0}^{\infty} \int_0^x s^n \, ds \\
= -\sum_{n=0}^{\infty} \frac{x^n}{n} \, ds
\]

For then by properties of logs,

\[ f(x) = \ln(7) \left( 1 - \frac{1}{7} x \right) = \ln(7) - \left( 1 - \frac{1}{7} x \right) \]

so that

\[ f(x) = \ln(7) - \sum_{n=1}^{\infty} \frac{x^n}{n!} \]

### 2 Some facts that may be helpful

#### 2.1

For a constant \( c > 0 \)

\[ \lim_{n \to \infty} \sqrt[n]{c} = \lim_{n \to \infty} c^{1/n} = 1 \]

#### 2.2

\[ \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n = e^x \]
\[
\lim_{n \to \infty} \left(1 - \frac{x}{n}\right)^n = e^{-x}
\]

For any fixed number \(x\).

2.2

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1
\]

\[
\lim_{x \to 0} \frac{\tan x}{x} = 1
\]

So you will have, for example,

\[
\lim_{n \to \infty} \frac{\sin \left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = 1
\]

2.3

When \(n\) is large,

\[\ln(n) \ll n \ll e^n \ll n! \ll n^n\]

(You can replace \(\ln(n)\) by any \(\log_a(n)\) \((a > 1)\) and replace \(e^n\) by any \(a^n\), \(a > 1\).

For example, you can say \(\log_2(n) < n < (1.01)^n\). )

2.4

\[(n + 1)! = (n + 1) \cdot n!\]

\[(2(n + 1))! = (2n + 2) \cdot (2n + 1) \cdot (2n) \cdots 1 = (2n + 2) \cdot (2n + 1) \cdot (2n)!\]

Note:

\[(2n)! \neq 2 \cdot n!\]

2.5

Be familiar with properties of exponential functions and logarithms.