In *An example of neckpinching for Ricci flow on* $\mathcal{S}^{n+1}$, we considered evolving metrics on $(-1,1)\times\mathcal{S}^n$ of the form

$$g = \varphi(x,t)^2(\mathrm{d}x)^2 + \psi(x,t)^2 g_{\mathcal{S}^n}$$

and imposed boundary conditions to ensure that these extend to smooth metrics on $\mathcal{S}^{n+1}$. In studying the evolution of these metrics by Ricci flow, we found it convenient to introduce the arc-length coordinate $s(x,t)=\int_0^x \varphi(x',t)\,\mathrm{d}x'$ and to write $\psi(s,t)$ as shorthand for $\psi\big(s(x,t),t\big)$.

In §10 of the paper, we presented an argument to prove that a particular class of solutions exhibits "single-point pinching" behavior. The argument proceeded by applying the maximum principle to a family of lower barriers for the quantity $v=\psi_s$ in order to show that ψ is strictly positive at any small positive distance from the singularity. Nataša Šešum has pointed out that we did not fully verify the boundary conditions when applying the maximum principle. In this short note, we present one possible approach to completing the proof. In order to keep the note relatively self-contained, we duplicate some results from the original paper in what follows.

The set-up

We have a solution of the system

$$\psi_t = \psi_{ss} - (n-1)\frac{1-\psi_s^2}{\psi}, \tag{1a}$$

$$\varphi_t = n\frac{\psi_{ss}}{\psi}\varphi, \tag{1b}$$

defined for $-1\le x\le 1$ and $0\le t<T$ originating from initial data that are reflection symmetric with exactly one "bump" per hemisphere. Thus:

$$\psi(-x,t)=\psi(x,t) \quad \text{and} \quad \varphi(-x,t)=\varphi(x,t) \quad \forall x,t, \tag{2}$$

$$\exists x_0\in(0,1) \text{ such that } \begin{cases}\psi_x(x,0)>0 & \text{on } (0,x_0),\\ \psi_x(x,0)<0 & \text{on } (x_0,1].\end{cases} \tag{3}$$

It follows from the Sturmian theorem (see Lemma 5.5) that there exists a smooth function $x_*:[0,T)\to(0,1)$ such that $\psi_x(x_*(t),t)=0$, while $\psi_x(x,t)$ is positive for $0<x<x_*(t)$ and negative for $x_*(t)<x\le 1$.

We may assume that our solution also has the following properties:

$$|\psi_s(x,t)|\le 1 \text{ for all } (x,t)\in(-1,1)\times[0,T) \qquad \text{(see Proposition 5.1)}, \tag{4}$$

$$\psi_t(x,t)\le 0 \text{ for all } (x,t)\in(-1,1)\times[0,T) \qquad \text{(see Proposition 5.2)}, \tag{5}$$

$$\sup_{x,t}\big|\psi\psi_{ss}\big|<\infty \qquad \text{(see Corollary 3.2)}. \tag{6}$$

As shown in Proposition 5.3, this implies the existence of C_0 such that $\big|(\psi^2)_t\big|\le C_0$ for all $x\in[-1,1]$ and $t\in[0,T)$. Then as shown in Proposition 5.4, the limit $\psi(x,T)=\lim_{t\nearrow T}\psi(x,t)$ exists, and the convergence is uniform for $|x|\le 1$.

The fix

Let $\Psi := \max_{0<x<1} \psi(x,T) > 0$. The assumption of one bump per hemisphere in our initial data implies that there exist $0 \le x_1 < x_3 \le x_4 < x_2 \le 1$ such that

$$\begin{aligned}
\psi(x,T) &= 0 && \forall x \in [0,x_1] \cup [x_2,1],\\
\psi(x,T) &= \Psi && \forall x \in [x_3,x_4],\\
0 < \psi(x,T) &< \Psi && \forall x \in (x_1,x_3) \cup (x_4,x_2),\\
\psi_s(x,T) &\ge 0 && \forall x \in [x_1,x_3],\\
\psi_s(x,T) &\le 0 && \forall x \in [x_4,x_2].
\end{aligned}$$

Lemma. *For all $x \in (x_1,x_2)$, the limit*

$$\varphi(x,T) = \lim_{t \nearrow T} \varphi(x,t)$$

exists and satisfies $\varphi(x,T) > 0$. The convergence is uniform on compact subsets of (x_1,x_2), and $\varphi(x,T)$ is continuous on (x_1,x_2).

Proof. We have

$$\frac{\partial}{\partial t} \log \varphi(x,t) = n \frac{\psi_{ss}}{\psi}.$$

By (6), $|\psi\psi_{ss}|$ is uniformly bounded. So by (5), we have

$$\left| \frac{\partial}{\partial t} \log \varphi(x,t) \right| \le \frac{C}{\psi(x,t)^2} \le \frac{C}{\psi(x,T)^2}.$$

Continuity of $\psi(x,T)$ implies $\inf_I \psi(x,T) > 0$ for any closed interval $I \subset (x_1,x_2)$, which in turn implies that $\sup_I \left|(\log\varphi)_t\right| < \infty$. Hence $\lim_{t\nearrow T} \log\varphi(x,t)$ exists uniformly for all $x \in I$. Upon exponentiation, we see that $\varphi(x,t)$ converges to a positive limit for all $x \in (x_1,x_2)$, and that the convergence is uniform on compact intervals $I \subset (x_1,x_2)$. □

Lemma. *For any $x_5 \in (x_1,x_3)$, there exists a time $t_0 \in (0,T)$ such that*

$$\inf_{t_0<t<T} \psi_s(x_5,t) > 0.$$

Proof. Choose $r > 0$ so small that $[x_5 - r, x_5 + r] \subset (x_1,x_3)$. On the interval (x_1,x_3), we know that $\psi_s(x,T) \ge 0$. Since $\psi(x_5,T) < \Psi = \psi(x_3,T)$, and since $\psi(x,t) \to \psi(x,T)$ uniformly in x, it follows that if t is sufficiently close to T, the function $x \mapsto \psi(x,t)$ attains its maximum somewhere in the interval $[x_5 + r, 1]$. Thus we have $\psi_s(x,t) > 0$ on $(x_5 - r, x_5 + r) \times (t_0,T)$ for some $t_0 \in (0,T)$. □

By constructing a lower barrier, we will obtain a uniform lower bound.

Lemma. *$\psi_s(x_5,t)$ has a positive lower bound for $t_0 \le t \le T$.*

Proof. The function $v = \psi_s$ satisfies

$$v_t = v_{ss} + \left\{ \frac{n-2}{\psi}\psi_{ss} - \frac{n-1}{\psi^2}(1-\psi_s^2) \right\} v. \tag{7}$$

Since we have chosen $r > 0$ so small that $[x_5 - r, x_5 + r] \subset (x_1,x_3)$, we know that $\psi(x,t)$ is uniformly bounded from below on $[x_5 - r, x_5 + r]$. Since $|\psi_s| \le 1$ and $|\psi\psi_{ss}| \le C$ everywhere, we find that there exists a constant Q such that

$$\left| \frac{n-2}{\psi}\psi_{ss} - \frac{n-1}{\psi^2}(1-\psi_s^2) \right| \le Q$$

on $[x_5 - r, x_5 + r] \times (t_0, T)$. It follows that v satisfies

(8) $$v_t - v_{ss} + Qv > 0 \quad \text{on} \quad [x_5 - r, x_5 + r] \times (t_0, T).$$

Now consider the function

$$V(x,t) := e^{-\lambda t} \cos\big(\mu(t)\varsigma(x,t)\big),$$

in which $\lambda > 0$ is a constant and $\mu(t) > 0$ is a function, both of which we will choose below. The function $\varsigma(x,t)$ is the signed distance from x_5 to x at time t:

$$\varsigma(x,t) := \int_{x_5}^{x} \varphi(x', t)\, \mathrm{d}x'.$$

Note that by definition,

$$\varsigma_s = \frac{\varsigma_x}{\varphi} = 1.$$

Since $|\log \varphi(x,t)|$ is uniformly bounded on $[x_5 - r, x_5 + r] \times [0, T]$, there exists $c > 0$ such that

$$c|x - x_5| \le |\varsigma(x,t)| \le \frac{1}{c}|x - x_5| \qquad \text{for} \qquad x \in [x_5 - r, x_5 + r], t \in [0, T].$$

To show that V is a lower barrier, we compute

$$\begin{aligned} V_t - V_{ss} + QV &= (Q + \mu^2 - \lambda)V + e^{-\lambda t}\left\{\mu'\varsigma + \mu\varsigma_t\right\}\sin\big(\mu\varsigma\big) \\ &= e^{-\lambda t}\left\{(Q + \mu^2 - \lambda)\cos(\mu\varsigma) - \left\{\frac{\mu'}{\mu} + \frac{\varsigma_t}{\varsigma}\right\}\mu\varsigma\sin(\mu\varsigma)\right\}. \end{aligned}$$

Because $\varphi_t = n\frac{\psi_{ss}}{\psi}\varphi$, there exists C_1 such that

$$|\varsigma_t(x,t)| = \left|\int_{x_5}^{x} n\frac{\psi_{ss}}{\psi}\varphi\, \mathrm{d}x\right| \le \left|C\int_{x_5}^{x}\varphi\, \mathrm{d}x\right| = C_1|\varsigma(x,t)|.$$

We now set $\mu(t) = \mu_0 e^{2C_1 t}$, and choose μ_0 so large that

$$\left|\mu_0 e^{2C_1 t}\varsigma(x_5 \pm r, t)\right| > \frac{\pi}{2}$$

for all $t \in [t_0, T)$. Since $x \mapsto \varsigma(x,t)$ is strictly increasing, there exist continuous functions $x_6, x_7 : [t_0, T) \to \mathbb{R}$ with

$$x_5 - r < x_6(t) < x_5 < x_7(t) < x_5 + r$$

and

$$\mu(t)\varsigma(x_6(t), t) = -\frac{\pi}{2}, \qquad \mu(t)\varsigma(x_7(t), t) = \frac{\pi}{2}.$$

On the region

$$\mathcal{B} := \{(x,t) \mid t_0 < t < T,\ x_6(t) < x < x_7(t)\},$$

we have

$$\frac{\mu'}{\mu} + \frac{\varsigma_t}{\varsigma} \ge 2C_1 - C_1 > 0.$$

We next choose $\lambda = Q + \mu_0^2 e^{4C_1 T}$ so that

$$\lambda > Q + \mu(t)^2 \quad \text{for} \quad t_0 \le t < T.$$

In the region $\mathcal{B}$, one has $|\mu\varsigma| < \frac{\pi}{2}$, so that

$$V_t - V_{ss} + QV < 0,$$

while by (8), $v = \psi_s$ satisfies the opposite inequality. The same is true of αV for any $\alpha > 0$.

We conclude the argument by verifying that $\alpha V \leq v$ on the parabolic boundary of $\mathcal{B}$ if α is small enough, and then invoking the maximum principle. Indeed, if $x = x_6(t)$ or $x = x_7(t)$, then $\mu(t)\varsigma(x,t) = \pm\pi/2$, so that $V(x,t) = 0 < v(x,t)$. At $t = t_0$, we observe that $v(x,t_0)$ is a positive continuous function of $x \in [x_5 - r, x_5 + r]$, so that $\inf_{|x-x_5|\leq r} v(x,t) > 0$. This implies that for small enough $\alpha > 0$, one has $\alpha V(x,t_0) \leq v(x,t_0)$ for all $x \in \big[x_6(t_0), x_7(t_0)\big]$.

By the maximum principle, we have $\alpha V(x,t) \leq v(x,t)$ for all $(x,t) \in \mathcal{B}$, and in particular,

$$v(x_5,t) \geq \alpha V(x_5,t) = \alpha e^{-\lambda t} \geq \alpha e^{-\lambda T}.$$

In short, $v(x_5,t) = \psi_s(x_5,t)$ has a positive lower bound for $t_0 \leq t \leq T$. □

The conclusion

At this point, one can continue as in §10 of the original paper (specifically, Proposition 10.2) to conclude that for every $\delta > 0$, there exists $\varepsilon > 0$ such that for all $t \in (t_0, T)$ and $x \in (0, x_5)$, one has

$$v(x,t) \geq \varepsilon\big(s(x,t) - \rho(t) - \delta\big),$$

where

$$\rho(t) = n \int_{t_0}^{t} \int_0^{x_5} \frac{\psi_s(x,t)^2}{\psi(x,t)^2}\, \mathrm{d}x\, \mathrm{d}t.$$

Finally, we note that the assumption in §10 that the diameter of the solution remains bounded is unnecessary: all solutions under consideration have bounded diameter, as is proved in a subsequent paper (S.B. Angenent, D. Knopf, *Precise asymptotics of the Ricci flow neckpinch,* Comm. Anal. Geom. **15** (2007), no. 4, 773–844).